J. Appl. Maths Mechs Vol. 56, No. 5, pp. 725–735, 1992
Printed in Great Britain.

0021–8928/92 $24.00 + .00

OSCILLATION SHAPE CONTROL IN RESONANT SYSTEMS†

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(*Received* 9 *January* 1992)

A range of new perturbation theory problems is considered. A connection is established between different types of oscillation shape in configuration space and manifolds defined in phase space. A construction of bases on these manifolds is given, so that each basis unit vector defines one of the evolution forms of an oscillation shape under the influence of the perturbation. Algebraic properties of the local evolution basis are established. A classification of the perturbations is introduced according to the nature of the evolution induced on the oscillation shape. The control and stability problems for the oscillation shapes are solved.

Similar problems include, in particular, the problem of controlling waves in uniaxial and triaxial gyroscopes, based on the inertia effect for elastic waves [1, 2].

1. THE PROBLEM OF THE PERTURBATION OF AN OSCILLATION SHAPE IN SYSTEMS WITH MULTIPLE ROOTS

A QUASILINEAR oscillatory system of the form

$$Aq^{\cdot\cdot} + Bq = \epsilon Q(t, q, q^{\cdot}) \tag{1.1}$$

is considered, where A and B are symmetric positive-definite $n \times n$ matrices, q is a vector of dimensions n, and the right-hand side is a perturbation which is formalized by the presence of the small factor ϵ.

Without loss of generality the matrix A can be taken to be the unit matrix, and the matrix B to be diagonal.

If there is no perturbation ($\epsilon = 0$), any trajectory of system (1.1) in configuration space is everywhere dense in a multidimensional parallelepiped, if the eigenfrequencies are incommensurable (the non-resonant case). If however there is resonance, a subspace exists in which every trajectory is a closed curve. Such curves are called Lissajous figures. They are unstable with respect to infinitesimally small perturbations: either they disappear completely, or they change to a figure with a different shape.

The following three problems are posed concerning these unstable trajectories: (1) to give a description of the evolution which these figures undergo when perturbations appear; (2) to classify external perturbation forces according to the evolution they create; and (3) to construct a control that ensures the stability of given figures.

We shall only consider principal resonances, i.e. resonances of smallest order for a given multiplicity. This corresponds to the presence of multiple frequencies for natural oscillations in system (1.1) for $\epsilon = 0$, for which it has a proper subspace corresponding to a multiple root, in which all trajectories are ellipses lying in some linear two-dimensional subspace. In the case of multiplicity of two, this manifold coincides with the proper subspace itself.

Only four fundamental types of infinitesimal evolution exist for this ellipse under perturbations: a change in the principle axes, a change in the orientation of the axes of the ellipse with respect to the basis space (we shall call this shape precession), a change in the velocity of motion of the point along

† *Prikl. Mat. Mekh.* Vol. 56, No. 5, pp. 827–836, 1992.

the ellipse, and finally, the transformation of the ellipse into a shape which cannot be considered as an evolution of one of the first three types. (We shall call this latter type of evolution shape destruction.)

The geometrical and algebraic properties of the given types of evolution depend principally on the dimensions of the eigenspaces in which this evolution proceeds, and hence they should be studied for every value of the multiplicity of the eigenfrequencies separately. Below we shall consider the cases $k = 2$ and $k = 3$.

2. THE CASE OF A DOUBLE EIGENFREQUENCY

In this case one can isolate the following subsystem from system (1.1)

$$q_i^{\cdot\cdot} + q_i = \epsilon Q_i(t, q, q^{\cdot}), \quad i = 1, 2 \tag{2.1}$$

in which we can make the change of phase variables $(q_1, q_2, q^{\cdot}_1, q^{\cdot}_2) \to (x_1, x_2, x_3, x_4)$ according to the formulae

$$q = (E\cos t,\ E\sin t)x, \quad q^{\cdot} = (-E\sin t,\ E\cos t)x \tag{2.2}$$

System (2.1) is rewritten in standard form to which we apply the averaging method [3]. In the first approximation of this method in slow variables we obtain the system

$$\tag{2.3}$$

the right-hand side of which is related as follows with the right-hand side of system (2.1)

$$X(x) = \frac{1}{2\pi} \int_0^{2\pi} \begin{Vmatrix} -E\sin t \\ E\cos t \end{Vmatrix} \begin{Vmatrix} Q_1 \\ Q_2 \end{Vmatrix} dt \tag{2.4}$$

E being the 2×2 unit matrix.

System (2.1) is connected to all the remaining equations of system (1.1). This connection in Eqs (2.3) will vanish if we assume that the dependence of Q_1 and Q_2 on t is periodic with period 2π or else is completely absent, and that their dependence on q and $q^{\cdot}$ does not contain powers higher than two. (The general case can be considered by introducing a small-scale variation of the variables q with subsequent construction of higher approximations by an averaging method.)

If $\epsilon = 0$, then in Eqs (2.3) $x \equiv \text{const}$ and in configuration space the trajectories of the original system are ellipses, i.e. every point of phase space x determines a specific elliptic trajectory in the space q. Among the elliptic trajectories there are degenerate trajectories of two types: these are either sections of a straight line or circles.

In the first case, x in (2.2) should satisfy the condition

$$K = \det \begin{Vmatrix} x_1 & x_3 \\ x_2 & x_4 \end{Vmatrix} = 0 \tag{2.5}$$

In the second case, relations (2.2) should define a rotation group, for which it is necessary and sufficient that $x_1 = x_4$ and $x_2 = -x_3$, or $x_1 = -x_4$ and $x_2 = x_3$.

Both the latter conditions can be combined

$$L = (x_1 \pm x_4)^2 + (x_2 \mp x_3)^2 = 0 \tag{2.6}$$

3. MANIFOLDS OF DEGENERATE SHAPES. LOCAL EVOLUTION BASIS

In the space of x, Eq. (2.5) defines a three-dimensional conic surface, every point of which corresponds to a straight-line oscillation in configuration space. Equation (2.6) defines a two-dimensional manifold, which is the "axis" of this cone, and each point of which corresponds to motion in a circle. All other points define elliptic trajectories.

In system (2.1) we choose initial conditions so that for $\epsilon = 0$ a straight-line oscillation shape will ensue, i.e. the initial point in system (2.3) lies on the cone (2.5). Then for $\epsilon \neq 0$ the straight-line oscillation shape will undergo one of the following evolutions: shape destruction, shape precession, change of oscillation frequency, or change of amplitude. To all these types of evolution of the straight-line shape there correspond definite phase space directions.

The direction of fastest destruction of the straight-line shape is given by the normal to the cone (2.5)

$$e_1 = dK/dx = \{x_4, -x_3, -x_2, x_1\} \tag{3.1}$$

To construct the directions giving precession, we subject (2.2) to a rotation transformation (A is a rotation matrix)

$$\{A\cos t,\ A\sin t\}x = \{E\cos t,\ E\sin t\}y \Rightarrow y = y(x, \alpha)$$

We find the relation between the new variables y and the old variables x together with the angle of rotation α

$$y_m = x_m \cos\alpha + x_{m+1}\sin\alpha, \quad y_{m+1} = -x_m \sin\alpha + x_{m+1}\cos\alpha, \quad m = 1, 3$$

The vector determining the required direction has the form

$$e_2 = dy/d\alpha|_{\alpha=0} = \{x_2, -x_1, x_4, -x_3\} \tag{3.2}$$

To construct the direction giving a change in frequency, we subject (2.2) to a translation transformation in time (τ is the transformation parameter)

$$\{E\cos(t+\tau),\quad E\sin(t+\tau)\}x = \{E\cos t,\ E\sin t\}y \Rightarrow y = y(x, \tau)$$

From this we also find the relation between the new variables and the old ones together with the translation parameter

$$y_m = x_m\cos\tau + x_{m+2}\sin\tau, \quad y_{m+2} = -x_m\sin\tau + x_{m+2}\cos\tau, \quad m = 1, 2$$

The vector giving the required direction is found to be

$$e_3 = dy/d\tau|_{\tau=0} = \{x_3, x_4, -x_1, -x_2\} \tag{3.3}$$

To construct the direction giving the change in amplitude, we make (2.2) undergo a dilation transformation

$$(1+\mu)\{E\cos t,\ E\sin t\}x = \{E\cos t,\ E\sin t\}y \Rightarrow y = y(x, \mu)$$

The required direction has the form

$$e_4 = dy/d\mu = \{x_1, x_2, x_3, x_4\} \tag{3.4}$$

The four vector fields (3.1)–(3.4) define a local basis for infinitesimal evolutions at every point of the vector space.

4. PROPERTIES OF THE EVOLUTION BASIS

We compute the Gram matrix [4] of the vector system (3.1)–(3.4)

$$G = \begin{Vmatrix} (e_1\cdot e_1)\dots(e_1\cdot e_4) \\ \vdots \qquad\qquad \vdots \\ (e_4\cdot e_1)\dots(e_4\cdot e_4) \end{Vmatrix} = \begin{Vmatrix} x^2 & 0 & 0 & 2K \\ 0 & x^2 & -2K & 0 \\ 0 & -2K & x^2 & 0 \\ 2K & 0 & 0 & x^2 \end{Vmatrix} \tag{4.1}$$

If follows from this that the evolution basis is orthogonal on the cone $K=0$. The Gram determinant

$$\det G = (x^4 - 4K\ )^2$$

is equal to zero on the manifold $2K = \pm x^2$, coinciding with the manifold (2.6), i.e. with the axis of the cone $K=0$. The four vector fields (3.1)–(3.4) generate four one-parameter Lie groups [3, 4] of mappings of the phase space into itself with operators

$$U_1 = x_4 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4}, \quad U_2 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \tag{4.2}$$

$$U_3 = x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} - x_2 \frac{\partial}{\partial x_4}, \quad U_4 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}$$

Computing the commutators, we find $[U_k, U_l] = 0$ for all k and l. This means that the evolution basis generates a four-parameter commutative (Abelian) Lie group, whose representation in the R^4 group of automorphisms is identical with the maximum commutative subgroup of the group $\mathrm{GL}(4, R)$. It follows from this that the general solution of system (2.3) for a right-hand side of the following form

$$x^{\cdot} = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 \tag{4.3}$$

where the a_k are arbitrary constants, is a composition of the general solutions of the following systems

$$x^{\cdot} = a_k e_k \quad k = 1, \ldots, 4$$

This means that the global evolutions of the oscillation shape in system (2.1), generated by the fields (3.1)–(3.4), do not depend on one another.

We shall follow as an example the evolution of a cone of rectilinear oscillations along the vector field e_1, (the cone K being invariant under the remaining fields).

Because $U_1 K = \|x\|^2$ and $U_1 \|x\|^2 = 4K$, the complete evolution of the cone along the group U_1 is

$$\exp(\pm U_1 \tau) K = K \operatorname{ch} 2\tau \pm \tfrac{1}{2} \|x\|^2 \operatorname{sh} 2\tau$$

As $\tau \to \infty$ the limiting manifold

$$K \pm \tfrac{1}{2} \|x\|^2 = 0$$

coincides with the axis of the cone K—(2.6).

Figure 1 shows the trajectory of a point generated by the field e_1.

5. CLASSIFICATION OF FORCES BY THE TYPE OF EVOLUTION THEY GENERATE

We consider in system (2.1) forces that are linear in the coordinates and velocities and independent of time

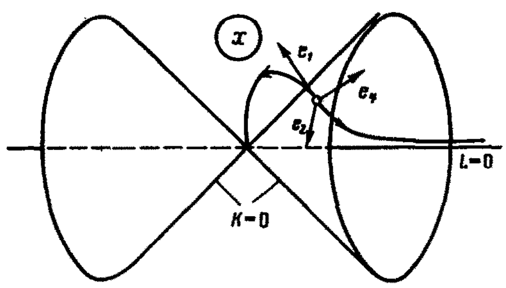

FIG. 1.

$$\begin{Vmatrix} Q_1 \\ Q_2 \end{Vmatrix} = (S + H + N) \begin{Vmatrix} q_1 \\ q_2 \end{Vmatrix} + (D + R + \Gamma) \begin{Vmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Vmatrix} \tag{5.1}$$

$$S = \begin{Vmatrix} s & 0 \\ 0 & s \end{Vmatrix}, \quad H = \begin{Vmatrix} h_1 & h_2 \\ h_2 & -h_1 \end{Vmatrix}, \quad N = \begin{Vmatrix} 0 & n \\ -n & 0 \end{Vmatrix}, \quad D = \begin{Vmatrix} d & 0 \\ 0 & d \end{Vmatrix}, \quad R = \begin{Vmatrix} r_1 & r_2 \\ r_2 & -r_1 \end{Vmatrix},$$

$$\Gamma = \begin{Vmatrix} 0 & \gamma \\ -\gamma & 0 \end{Vmatrix}$$

Here S is a symmetric matrix of potential position forces of spherical type, H is a similar matrix of hyberbolic type, N is a skew-symmetric matrix of circular (properly non-conservative) position forces, D is a symmetric matrix of dissipative forces of spherical type, R is a similar matrix of hyperbolic type, and Γ is a skew-symmetric matrix of gyroscopic forces. The matrices S and D are diagonal (spherical tensors), and the matrices H and R have zero trace (deviators).

Substituting (5.1) into (2.4), we find expressions for the right-hand sides in system (2.3) corresponding to the following forces

$$S\colon\ X(x) = \tfrac{1}{2} s\{-x_3,\ -x_4,\ x_1,\ x_2\}, \quad H\colon\ X(x) = \tfrac{1}{2} h_1\{-x_3,\ x_4,\ x_1,\ -x_2\} +$$
$$+ \tfrac{1}{2} h_2\{-x_4,\ -x_3,\ x_2,\ x_1\}, \quad N\colon X(x) = \tfrac{1}{2} n\{-x_4,\ x_3,\ x_2,\ -x_1\},$$
$$D\colon\ X(x) = \tfrac{1}{2} d\{x_1,\ x_2,\ x_3,\ x_4\}, \quad R\colon\ X(x) = \tfrac{1}{2} r_1\{x_1,\ -x_2,\ x_3,\ -x_4\} +$$
$$+ \tfrac{1}{2} r_2\{x_2,\ x_1,\ x_4,\ x_3\}, \quad \Gamma\colon\ X(x) = \tfrac{1}{2}\gamma\{x_2,\ -x_1,\ x_4,\ -x_3\}$$

In order to clarify what evolution is produced by all these forces, it is sufficient to project them onto the vectors of the evolution basis (3.1)–(3.4). The results are presented in Table 1, which gives a complete picture of the influence of linear perturbations on the evolution of the oscillation shapes.

For example, spherical potential forces (S) only lead to a change in the oscillation frequency if the oscillation shape is rectilinear ($K = 0$). If however $K \neq 0$ (elliptic trajectory) then there also appears a precession for the ellipse. Circular forces only lead to the destruction of the rectilinear shape ($K = 0$) and to a change in the amplitude if $K \neq 0$.

6. THE PROBLEM OF STABILIZING THE SHAPE OF THE OSCILLATION

In linear systems of the form (2.1) the rectilinear shape of the oscillations is unstable for constantly-acting perturbations. As follows from Table 1, in the presence of forces H and N of no matter how small an amplitude, this shape is impossible. We pose the following problem: it is required to find a form of feedback such that the rectilinear shape is asymptotically stable. Here the feedback should be such that it does not lead to any other kind of shape evolution.

This problem has technical applications [1]. If the rectilinear shape is stable, then observation of its precession gives information about any gyroscopic forces present in the system.

We apply forces $Q(x)$ to system (2.1) such that the cone (2.5) is a stable integral manifold [6]

TABLE 1

	S	H	N	D	R	Γ
e_1	0	$-h_1K' + \tfrac{1}{2}h_2K''$	$-\tfrac{1}{2}\Vert x\Vert^2 n$	dK	0	0
e_2	sK	0	0	0	$r_1K' - \tfrac{1}{2}r_2K''$	$\tfrac{1}{2}\gamma\Vert x\Vert^2$
e_3	$-\tfrac{1}{2}s\Vert x\Vert^2$	$-\tfrac{1}{2}(h_1K'' + h_2K)$	0	0	0	$-\gamma K$
e_4	0	-0	$-nK$	$\tfrac{1}{2}d\Vert x\Vert^2$	$\tfrac{1}{2}r_1K'' + r_2K'$	0

$K' = x_1x_2 + x_3x_4,\quad K'' = x_1^2 - x_2^2 + x_3^2 - x_4^2$

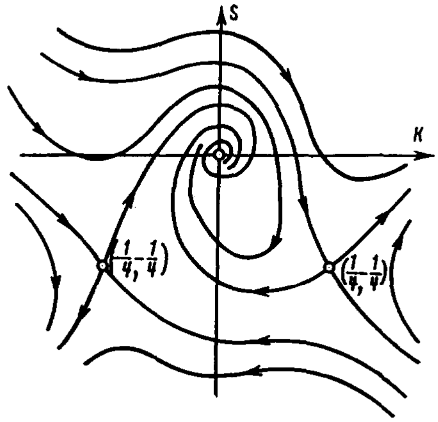

FIG. 2.

under constantly acting perturbations. In order for these forces not to influence the frequency and precession of the shape, they should satisfy the conditions $(Q \cdot e_2) \equiv 0$ and $(Q \cdot e_3) \equiv 0$.

In system (2.3) we choose a feedback (control) of the form

$$X(x) = -Ke_1 - Se_4, \quad S = \tfrac{1}{2}(\|x\|^2 - 1) \tag{6.1}$$

If $S = 0$ the oscillation amplitude is unity.

The presence in the control of a term proportional to S pursues the aim of not only stabilizing the shape itself, but also its amplitude. Equations (2.1) take the form

$$dx/dt = -\epsilon(Ke_1 + Se_4) \tag{6.2}$$

We construct equations governing the changes of K and S in the force (6.2). Using the fact that $dK/dt = dK/dx \cdot x^{\bullet} = e_1 x^{\bullet}$ and $dS/dt = dS/dx \cdot x^{\bullet} = e_4 x^{\bullet}$, and also bearing in mind the Gram matrix (4.1) and the equality $\|x\|^2 = 2S + 1$, we obtain

$$dK/dt = -K(1 + 2S) - 2SK, \quad dS/dt = -2K^2 - S(1 + 2S).$$

The phase portrait of this two-dimensional system is shown in Fig. 2. We have two singular points $(-\tfrac{1}{4}, -\tfrac{1}{4})$ and $(\tfrac{1}{4}, -\tfrac{1}{4})$ of saddle-point type and one singular point $(0, 0)$ of stable-focus type. Because $K \cap S$ is compact it follows from the exponential stability of the zero point that $K \cap S$ is stable under constantly acting perturbations.

In order to ensure a control of the form (6.1) in system (2.3), it is necessary to apply non-conservative forces with matrices N and D to system (2.1), as can be found from Table 1

$$q^{\cdot\cdot} + q = -\epsilon(Nq + Dq^{\cdot}) \tag{6.3}$$

and to choose numbers n and d proportional to K and S.

Then small perturbations to system (6.3) do not lead to the destruction of the rectilinear shape of the oscillations, or to changes to its amplitude. If however the initial shape of the oscillation is not rectilinear, then by Eq. (6.3) it will tend asymptotically to rectilinear.

7. THE CASE OF A TRIPLE NATURAL FREQUENCY (1:1:1 RESONANCE)

In this case one can isolate the following subsystem from system (1.1):

$$q_i^{\cdot\cdot} + q_i = \epsilon Q_i(t, q, q^{\cdot}), \quad i = 1, 2, 3 \tag{7.1}$$

As in the preceding case and with the same conditions, to a first approximation of the normal form, there is no coupling between this subsystem and the others. We write the general solution of this system for $\epsilon = 0$ as follows:

$$q = (E\cos t, E\sin t)x, \qquad q = \{q_1, q_2, q_3\} \tag{7.2}$$

where the factor next to x is a 3×6 matrix (E being the unit 3×3 matrix), and x is a six-dimensional vector of arbitrary constants.

8. OSCILLATION SHAPES AND THEIR PHASE-SPACE IMAGES

Formula (7.2) defines an elliptic plane trajectory in configuration space. It establishes a one-to-one correspondence between all elliptic trajectories in system (7.1) for $\epsilon = 0$ and all points in the phase space $x \in R^6$. The plane in which the ellipse lies is found from the condition

$$\det \begin{Vmatrix} q_1 & x_1 & x_4 \\ q_2 & x_2 & x_5 \\ q_3 & x_3 & x_6 \end{Vmatrix} = 0$$

We introduce the following notation for the coefficients of the normal to the plane

$$K_1 = x_3x_4 - x_1x_6, \qquad K_2 = x_1x_5 - x_2x_4, \qquad K_3 = x_2x_6 - x_3x_5 \tag{8.1}$$

When the elliptic trajectory degenerates into a section of a straight line, we have

$$K = K_1^2 + K_2^2 + K_3^2 = 0 \tag{8.2}$$

Equation (8.2) defines a three-dimensional cone in the space of x, with a one-to-one relation with all rectilinear oscillations in system (7.1) when $\epsilon = 0$.

9. FIRST APPROXIMATION TO THE NORMAL FORM

Making the change of variables $(q, q^{\bullet}) \to (x)$ in (7.1) according to the formulae

$$q = (E\cos t, E\sin t)x, \qquad q^{\bullet} = (-E\sin t, E\cos t)x$$

(where E is the unit 3×3 matrix) and averaging, we obtain a system similar to (2.3) and (2.4).

10. BASIS OF INFINITESIMAL EVOLUTIONS

This basis is constructed in the same ways as the case of 1 : 1 resonance, but has its own features in the case considered.

We have three directions for destroying the rectilinear shape

$$e_1 = dK_1/dx = \{0, x_6, -x_5, 0, -x_3, x_2\}, \qquad e_2 = dK_2/dx = \{-x_6, 0, x_4, x_3, 0, -x_1\},$$
$$e_3 = dK_3/dx = \{x_5, -x_4, 0, -x_2, x_1, 0\} \tag{10.1}$$

There are also three directions for spatial precession. Suppose A is an orthogonal matrix, $A^{\mathrm{T}} = A^{-1}$:

$$\{A\cos t, A\sin t\}x = \{E\cos t, E\sin t\}y \Rightarrow y = y(x, \alpha, \beta, \gamma)$$

where α, β and γ are local coordinates of the rotation A (Krylov angles). Then

$$\begin{aligned} &e_4 = dy/d\alpha = \{-x_2, x_1, 0, -x_5, x_4, 0\}, \qquad e_5 = dy/d\beta = \{x_3, 0, -x_1, x_6, 0, -x_4\}, \\ &e_6 = dy/d\gamma = \{0, -x_3, x_2, 0, -x_6, x_5\} \qquad \text{for} \qquad \alpha = \beta = \gamma = 0 \end{aligned} \tag{10.2}$$

The directions of the change of frequency and amplitude are one-dimensional, as in the preceding case

$$\{E\cos(t+\tau), E\sin(t+\tau)\}x = \{E\cos t, E\sin t\}y \Rightarrow y = y(x,\tau)$$

$$e_7 = dy/d\tau\,|_{\tau=0} = \{x_4, x_5, x_6, -x_1 - x_2, -x_3\} \tag{10.3}$$

$$(1+\mu)\{E\cos t, E\sin t\}x = \{E\cos t, E\sin t\}y \Rightarrow y = y(x,\mu)$$

$$e_8 = dy/d\mu\,|_{\mu=0} = \{x_1, x_2, x_3, x_4, x_5, x_6\} \tag{10.4}$$

11. PROPERTIES OF THE BASIS

The Gram matrix of the eight vectors (10.1)–(10.4) has the block form

$$\begin{Vmatrix} G_0 & G_2 & G_3 & G_4 \\ G_2 & G_0 & G_5 & G_3 \\ G_3 & G_5 & G_1 & G_6 \\ G_4 & G_3 & G_6 & G_1 \end{Vmatrix} \tag{11.1}$$

where G_0, the (3×3) Gram matrix of the destruction bundle, is identical with the Gram matrix of the precession bundle. Its determinant is equal to

$$\det G_0 = K\|x\|^2 \tag{11.2}$$

The remaining matrices have the form

$$G_2 = \begin{Vmatrix} K_3 & 0 & -K_2 \\ 0 & -K_3 & K_1 \\ -K_1 & K_2 & 0 \end{Vmatrix}, \quad G_3 = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix}, \quad G_4 = 2\begin{Vmatrix} K_1 \\ K_2 \\ K_3 \end{Vmatrix}, \quad G_5 = 2\begin{Vmatrix} K_2 \\ K_1 \\ K_3 \end{Vmatrix},$$

$$G_1 = \|x\|^2, \quad G_6 = 0$$

We note the following properties of the basis.

1. On the $K=0$ cone and $\det G_0 = 0$, i.e. the vectors e_1, e_2, e_3 are linearly dependent on the cone. Similarly for the vectors e_4, e_5, e_6.

2. On the $K=0$ cone there are orthogonal subspaces

$$e_1, e_2, e_3 \perp e_4, e_5, e_6 \perp e_7 \perp e_8$$

3. The algebra of the evolution operators generated by the vector fields (10.1)–(10.4) is eight-dimensional and non-commutative. In the preceding (1 : 1) case the dimensions of this algebra were the same as the dimensions of the phase space and it was Abelian.

4. The $K=0$ is an invariant manifold of the precession, frequency-change and amplitude-change subgroups, i.e. e_4, e_5, e_6, e_7, e_8 are tangential to the cone.

5. The evolution basis (10.1)–(10.4) is non-holonomic. Apart from the destruction vector fields (10.1), which are potential, only e_8 is a potential field

$$e_8 = dS/dx, \qquad S = \tfrac{1}{2}(\|x\|^2 - 1) \tag{11.3}$$

12. CLASSIFICATION OF THE PERTURBATIONS

We will represent the right-hand sides of (7.1) in the same form as in (5.1), taking account of the different dimensions. Because the effects of each of the component forces of (5.1) can be analysed

TABLE 2

	S	H	N	D	R	Γ
e_1, e_2, e_3	0	+	+	0	0	0
e_4, e_5, e_6	0	0	0	0	+	+
e_7	+	+	0	0	0	0
e_8	0	0	0	+	+	0

independently and the left-hand side of system (7.1) is invariant under rotation of the axes, each of the matrices in relation (5.1) in its 3×3 version can be assumed to be in its canonical form. Substituting them into the operator (9.1) and projecting the result along the vectors (10.1)–(10.4), we arrive at Table 2, constructed for the case $K=0$. The plus signs signify that the corresponding projection is non-zero. Comparison of Tables 1 and 2 shows their qualitative equivalence.

13. CONTROL OF THE OSCILLATION SHAPE

In order to ensure the asymptotic stability of the rectilinear oscillation shape in system (7.1) for $\epsilon = 0$, we introduce feedback into the right-hand side of system (7.1), formed, as in (6.1), as follows:

$$x^{\bullet} = -\sum_{i,j} a_{ij} e_i K_j - e_8 S \quad (i, j = 1, 2, 3) \tag{13.1}$$

In this system the $x^{\bullet} \equiv 0$ steady-state is reached on the manifold $K=0$, $S=0$. The coefficient matrix a_{ij} should be chosen so that being bounded on the compact set $K=0$, $S=0$, it ensures the asymptotic stability of the steady-state with the maximum degree of stability. We differentiate (8.1) invoking system (3.1)

$$dK_l/dt = dK_l/dx' = -\sum_{i,j} a_{ij}(e_l \cdot e_i) K_j - (e_l \cdot e_8) S \tag{13.2}$$

If the matrix $\{a_{ij}\}$ is chosen to be the inverse of the Gram matrix $G_0 = \{(e_i \cdot e_j)\}$, then using $(e_l \cdot e_8) = 2K_l$ we obtain

$$dK_l/dt = -K_l - 2K_l S \tag{13.3}$$

The degree of stability is a maximum, but because $\det G_0 = 0$ on the cone the matrix $\{a_{ij}\}$ is unbounded.

In order to avoid this it is sufficient to choose the matrix $\{a_{ij}\}$ to be equal to the matrix of cofactors of the matrix G_0: $\{a_{ij}\} = \{G_{0ij}\}$. Then instead of (13.3) we have

$$dK_l/dt = -K_l \det G_0 - 2K_l S$$

Computing the derivative of S and using (11.2), we obtain

$$dK/dt = -2K^2(1+2S) - 4KS, \quad dS/dt = -\sum_{i,j} G_{0ij} K_i K_j - S(1+2S) \tag{13.4}$$

Since G_{0ij} depends on x, here, unlike in the plane case (6.4), one cannot write system (13.4) only in variables defining the stabilized manifold —K, S. For closure one must attach system (13.1) to system (13.4).

To prove the stability of the manifold $K=S=0$ in system (13.1), (13.4) we construct comparison systems, for which we write upper and lower estimates for the quadratic form $\sum G_{0ij} K_i K_j$. Because of the positive definiteness of this form we have

$$\mu(x)(K_1^2 + K_2^2 + K_3^2) \leqslant \sum G_{0ij} K_i K_j \leqslant \lambda(x)(K_1^2 + K_2^2 + K_3^2)$$

where $\mu(x)$ and $\lambda(x)$ are the minimum and maximum eigenvalues of the matrix G_0. Since the $G_{0ij}(x)$ are homogeneous functions of the fourth degree, we have

$$\mu(x) = \| x \|^4 \mu(x'), \qquad \lambda(x) = \| x \|^4 \lambda(x') \qquad \text{где} \quad \| x' \| = 1$$

We introduce the notation

$$\nu_1 = \min_{\|x'\|=1} \mu(x'), \qquad \nu_2 = \max_{\|x'\|=1} \lambda(x')$$

As a result the quadratic form under consideration acquires the limits

$$\nu_1 (1 + 2S)^2 K \leqslant \Sigma\, G_{0ij} K_i K_j \leqslant \nu_2 (1 + 2S)^2 K$$

This gives us the following comparison systems

$$dK^i/dt = -2(K^i)^2 (1 + 2S^i) - 4K^i S^i, \quad dS^i/dt = -\nu_i (1 + 2S^i)^2 K^i - S^i (1 + 2S^i) \quad (i = 1, 2)$$

If $S(0) > 0$, then $i = 1$ and $K(t) \leqslant K^1(t)$, $S(t) \leqslant S^1(t)$
If $S(0) < 0$, then $i = 2$ and $K(t) \leqslant K^2(t)$, $S(t) \geqslant S^2(t)$

Tracing the resonant terms out of the comparison systems, we obtain their normal form

$$dK^i/dt = -2(K^i)^2, \qquad dS^i/dt = -S^i - 4\nu_i K^i S^i$$

from which it follows that $K^i \to 0$ and $S^i \to 0$.

Consequently, if $S(0) < 0$, then

$$0 \leqslant K(t) \leqslant K^2 \to 0, \qquad 0 > S(t) \geqslant S^2(t) \to 0$$

i.e. $K(t) \to 0$ and $S(t) \to 0$.

If $S(0) > 0$, then

$$0 \leqslant K(t) \leqslant K^1 \to 0, \qquad S(t) \leqslant S^1 \to 0$$

If $S(t)$ always remains positive here, it follows that $K(t) \to 0$ and $S(t) \to 0$. If however $S(t)$ changes sign, then starting from the instant at which $S = 0$ we have to change to the comparison system $i = 2$.

Thus it has been shown that the chosen control ensures the asymptotic stability of the manifold $K = S = 0$, but it is not now exponential, which is explained by the degeneracy of the destruction bundle on the cone.

14. CONCLUSIONS

We note the basic qualitative difference between the plane case (resonance 1:1) and the three-dimensional spatial case (resonance 1:1:1).

If in the plane (1:1) case the destruction direction for the rectilinear oscillation shape and the precession direction for that shape are one-dimensional in phase space, then in the (1:1:1) case the destruction and precession directions are given by a three-dimensional linear manifold, degenerating into two-dimensions on $K = 0$.

In the plane case one can choose a feedback which ensures exponential asymptotic stability of the manifold of rectilinear shapes, and therefore also their stability under constantly acting perturbations. In the three-dimensional case one can only ensure asymptotic stability of power form for that manifold. The question of its stabilization under constantly acting perturbations therefore remains open.

The basis of infinitesimal evolutions in the plane case has dimensions of four, identical with the dimensions of the phase space and generating a four-dimensional Abelian parametric Lie group in the latter. In the three-dimensional case this basis has dimensions of eight, which exceed by two the dimensions of the phase space in which it generates an eight-parameter non-commutative Lie group.

It is of interest to continue the consideration of resonant cases to not only higher multiplicities, but also to higher orders.

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Translated by R.L.Z.